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EXACT SOLUTIONS FOR A QUANTUM-MECHANICAL PARTICLE WITH SPIN 1 IN THE EXTERNAL HOMOGENEOUS MAGNETIC FIELD

With the use of the general covariant matrix 10-dimensional Petiau – Duffin – Kemmer formalism in cylindrical coordinates and tetrad there are constructed exact solutions of the quantum-mechanical equation for a particle with spin 1 in presence of an external homogeneous magnetic field. There are separated three linearly independent types of solutions; in each case the formula for energy levels has been found.

1 Introduction, setting the problem

The problem of a quantum-mechanical particle in the external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger's) non-relativistic particle with spin 0, and fermions (non-relativistic Pauli's and relativistic Dirac's) with spin 1/2 (the first investigation were [1, 2, 3, 4]). In the present paper, exact solutions for a vector particle with spin 1 will be constructed explicitly. The most popular quantum-mechanical problem for such a particle is that in presence of external Coulomb potential [4].

To treat the problem we take the matrix Petiau – Duffin – Kemmer approach in the theory of the vector particle extended to a general covariant form on the base of tetrad formalism (recent consideration and list of references see in [5, 6]).

The main equation in tetrad form is [6]

$$\left[i \beta^\alpha(x) (\partial_\alpha + B_\alpha - i \frac{e}{\hbar} A_\alpha) - \frac{Mc}{\hbar} \right] \Psi(x) = 0 ,$$

$$\beta^\alpha(x) = \beta^a e_{(a)}^\alpha(x), \quad B_\alpha(x) = \frac{1}{2} J^{ab} e_{(a)}^\beta \nabla_\alpha e_{(b)\beta} ; \quad (1)$$

$e_{(a)}^\alpha(x)$ is a tetrad, J^{ab} stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note Mc/\hbar as M). To the homogeneous magnetic field $\mathbf{B} = (0, 0, B)$ corresponds 4-potential

$$A^a = (0, \vec{A}) = (0, \frac{1}{2} \vec{B} \times \vec{r}) = \frac{B}{2} (0, -x^2, +x^1, 0) ;$$

in the cylindric coordinates it is given by a simple expression

$$(ct, r, \phi, z), \quad dS^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 ,$$

$$A_0 = 0, \quad A_r = 0, \quad A_\phi = -\frac{Br^2}{2}, \quad A_z = 0 . \quad (2)$$

Choosing a diagonal cylindric tetrad

$$e_{(0)}^\alpha = (1, 0, 0, 0), \quad e_{(1)}^\alpha = (0, 1, 0, 0), \quad e_{(2)}^\alpha = (0, 0, \frac{1}{r}, 0), \quad e_{(3)}^\alpha = (0, 0, 0, 1) . \quad (3)$$

after simple calculation, the main equation (1) is reduced to the form

$$\left[i\beta^0 \partial_0 + i\beta^1 \partial_r + i\frac{\beta^2}{r} (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + i\beta^3 \partial_z - M \right] \Psi(t, r, \phi, z) = 0. \quad (4)$$

For brevity we will note $(eB/2\hbar)$ as B . It is best to chose the matrices β^a in the so-called cyclic form, where the generator J^{12} has a diagonal structure. In block-form $(1 - 3 - 3 - 3)$ these matrices are

$$\beta^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \beta^i = \begin{vmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{vmatrix},$$

where e_i, τ_i denote

$$e_1 = \frac{1}{\sqrt{2}}(-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad e_3 = (0, i, 0),$$

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{vmatrix}, \quad \tau_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = s_3.$$

Entering eq. (4) generator J^{12} is given by

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 = -i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{vmatrix} = -i S_3.$$

2 Separation of the variables

With the use of special substitution (it corresponds to diagonalization of the third projections of momentum P_3 and angular momentum J_3 for a particle with spin 1)

$$\Psi = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix}, \left[\epsilon\beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r}(m + Br^2 - S_3) - k\beta^3 - M \right] \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix} = 0. \quad (5)$$

after calculations we arrive at the radial system of ten equations

$$\begin{aligned} -b_{m-1} E_1 - a_{m+1} E_3 - ik E_2 &= M\Phi_0, \\ -ib_{m-1} H_1 + ia_{m+1} H_3 + i\epsilon E_2 &= M\Phi_2, \\ ia_m H_2 + i\epsilon E_1 - k H_1 &= M\Phi_1, \\ -ib_m H_2 + i\epsilon E_3 + k H_3 &= M\Phi_3, \end{aligned} \quad (6)$$

$$\begin{aligned} a_m \Phi_0 - i\epsilon \Phi_1 &= ME_1, & -ia_m \Phi_2 + k \Phi_1 &= MH_1, \\ b_m \Phi_0 - i\epsilon \Phi_3 &= ME_3, & ib_m \Phi_2 - k \Phi_3 &= MH_3, \\ -i\epsilon \Phi_2 - ik \Phi_0 &= ME_2, & ib_{m-1} \Phi_1 - ia_{m+1} \Phi_3 &= MH_2, \end{aligned} \quad (7)$$

where special abbreviations are used for first order differential operators

$$\frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m + Br^2}{r}\right) = a_m, \quad \frac{1}{\sqrt{2}}\left(-\frac{d}{dr} + \frac{m + Br^2}{r}\right) = b_m.$$

From (6) – (7) it follow 4 equations for the components Φ_a

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 + i\epsilon (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) &= 0, \\ (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 - ik (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) &= 0, \\ (-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i\epsilon a_m \Phi_0 + ik a_m \Phi_2 &= 0, \\ (-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i\epsilon b_m \Phi_0 + ik b_m \Phi_2 &= 0; \end{aligned} \quad (8)$$

3 Special simple class of solutions

There exists a simple linear condition on 4-vector Φ_a , leading to a second order differential equation. Let it be $\Phi_1 = 0$, $\Phi_3 = 0$, the system (8) gives

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 &= 0, \\ (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 &= 0, \\ i a_m (\epsilon \Phi_0 + ik \Phi_2) = 0, \quad i b_m (\epsilon \Phi_0 + ik \Phi_2) &= 0. \end{aligned} \quad (9)$$

From two last equations in (9) we conclude that

$$\epsilon \Phi_0 + k \Phi_2 = 0 \quad (10)$$

correspondingly, the first two in (9) take the form

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2) \Phi_0 &= 0, \\ (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2) \Phi_2 &= 0. \end{aligned} \quad (11)$$

Because, we can readily get

$$-b_{m-1} a_m - a_{m+1} b_m = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} = \Delta,$$

eqs. (11) are differential equations of one the same type that is operative in the theory of a scalar particle in magnetic field

$$(\Delta + \epsilon^2 - k^2 - M^2) \Phi_0 = 0, \quad (\Delta + \epsilon^2 - k^2 - M^2) \Phi_2 = 0. \quad (12)$$

All the remaining component of the 10-dimensional function can be found straightforwardly as in accordance with the relations

$$\begin{aligned} \Phi_1 = 0, \quad \Phi_3 = 0, \quad \epsilon \Phi_0 + k \Phi_2 &= 0, \\ a_m \Phi_0 = ME_1, \quad a_m \Phi_2 = iMH_1, \quad b_m \Phi_0 = ME_3, \\ b_m \Phi_2 = -iMH_3, \quad (\epsilon \Phi_2 + k \Phi_0) = iME_2, \quad 0 = H_2. \end{aligned} \quad (13)$$

In general, there must exist three types of solutions for the particle with spin 1, we have found only one that.

4 General analysis of the radial equations

Eqs. (8) can be transformed to the form

$$\begin{aligned} & [-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2] (k \Phi_0 + \epsilon \Phi_2) = 0 , \\ & [-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2] (\epsilon \Phi_0 + k \Phi_2) = \\ & = (\epsilon^2 - k^2) [(\epsilon \Phi_0 + k \Phi_2) - (i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3)] ; \end{aligned} \quad (14)$$

$$\begin{aligned} & (-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i \epsilon a_m \Phi_0 + i k a_m \Phi_2 = 0 , \\ & (-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i \epsilon b_m \Phi_0 + i k b_m \Phi_2 = 0 . \end{aligned} \quad (15)$$

Let us introduce new variables

$$F(r) = k \Phi_0(r) + \epsilon \Phi_2(r) , \quad G(r) = \epsilon \Phi_0(r) + k \Phi_2(r) , \quad (16)$$

then eqs. (14) – (15) read

$$\begin{aligned} & [-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2] F = 0 , \\ & [-b_{m-1} a_m - a_{m+1} b_m - M^2] G = -(\epsilon^2 - k^2) (i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3) , \end{aligned} \quad (17)$$

$$\begin{aligned} & (-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i a_m G = 0 , \\ & (-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i b_m G = 0 . \end{aligned} \quad (18)$$

For two equations in (18), let us multiply the first (from the left) by b_{m-1} and the second by the a_{m+1} , which result in

$$\begin{aligned} & -b_{m-1} a_m (b_{m-1} \Phi_1) + (\epsilon^2 - k^2 - M^2) (b_{m-1} \Phi_1) + b_{m-1} a_m (a_{m+1} \Phi_3) + i b_{m-1} a_m G = 0 , \\ & -a_{m+1} b_m (a_{m+1} \Phi_3) + (\epsilon^2 - M^2 - k^2) (a_{m+1} \Phi_3) + a_{m+1} b_m (b_{m-1} \Phi_1) + i a_{m+1} b_m G = 0 . \end{aligned} \quad (19)$$

Again, let us introduce two new variables

$$b_{m-1} \Phi_1 = Z_1 , \quad a_{m+1} \Phi_3 = Z_3 ; \quad (20)$$

eqs. (19) read as follows

$$\begin{aligned} & -b_{m-1} a_m Z_1 + (\epsilon^2 - k^2 - M^2) Z_1 + b_{m-1} a_m Z_3 + i b_{m-1} a_m G = 0 , \\ & -a_{m+1} b_m Z_3 + (\epsilon^2 - M^2 - k^2) Z_3 + a_{m+1} b_m Z_1 + i a_{m+1} b_m G = 0 . \end{aligned} \quad (21)$$

With the aid of new functions $f(r), g(r)$

$$Z_1 = \frac{f+g}{2} , \quad Z_3 = \frac{f-g}{2} , \quad Z_1 + Z_3 = f , \quad Z_1 - Z_3 = g ; \quad (22)$$

the system (21) is transformed to the following ones

$$\begin{aligned} & -b_{m-1} a_m g + (\epsilon^2 - k^2 - M^2) \frac{f+g}{2} + i b_{m-1} a_m G = 0 , \\ & a_{m+1} b_m g + (\epsilon^2 - M^2 - k^2) \frac{f-g}{2} + i a_{m+1} b_m G = 0 . \end{aligned} \quad (23)$$

Combining these equations we get

$$\begin{aligned} [-b_{m-1}a_m - a_{m+1}b_m + \epsilon^2 - k^2 - M^2] g + i(b_{m-1}a_m - a_{m+1}b_m) G &= 0, \\ (-b_{m-1}a_m + a_{m+1}b_m) g + (\epsilon^2 - k^2 - M^2)f + i(b_{m-1}a_m + a_{m+1}b_m)G &= 0. \end{aligned} \quad (24)$$

In these variables, eqs. (17) can be written as

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1}b_m + \epsilon^2 - M^2 - k^2) F &= 0, \\ (-b_{m-1}a_m - a_{m+1}b_m - M^2) G &= -i(\epsilon^2 - k^2) f. \end{aligned} \quad (25)$$

Further, with the use of identities

$$-b_{m-1} a_m - a_{m+1}b_m = \Delta, \quad -b_{m-1} a_m + a_{m+1}b_m = 2B. \quad (26)$$

eqs. (25) and (24) can be written down as follows

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) F &= 0, \\ \Delta G &= M^2 G - i(\epsilon^2 - k^2) f, \\ (\Delta + \epsilon^2 - k^2 - M^2) g &= 2iB G, \\ (\epsilon^2 - k^2 - M^2) f - i\Delta G + 2B g &= 0. \end{aligned} \quad (27)$$

With the help of the second equation, from the forth one it follows the linear relationship

$$f = -i G + \frac{2B}{M^2} g. \quad (28)$$

Now, excluding the function f in the second one in (27)

$$(\Delta + \epsilon^2 - k^2 - M^2) G = -i(\epsilon^2 - k^2) \frac{2B}{M^2} g. \quad (29)$$

Thus, the general problem is reduced to the system of four equations

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) F &= 0, \\ f &= -i G + \frac{2B}{M^2} g, \\ (\Delta + \epsilon^2 - k^2 - M^2) g &= 2iB G, \\ (\Delta + \epsilon^2 - k^2 - M^2) G &= -2iB \frac{\epsilon^2 - k^2}{M^2} g, \end{aligned} \quad (30)$$

The structure of this system allows to separate an evident linearly independent solution as follows

$$\begin{aligned} f(r) &= 0, & g(r) &= 0, & H(r) &= 9, \\ F(r) &\neq 0, & (\Delta - k^2 - M^2 + \epsilon^2) F &= 0. \end{aligned} \quad (31)$$

corresponding functions and energy spectrum are known (also see below). We are to solve the system of two last equations in (30), in matrix form it reads (let $\gamma = (\epsilon^2 - k^2)/M^2$)

$$(\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g(r) \\ G(r) \end{vmatrix} = \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} \begin{vmatrix} g(r) \\ G(r) \end{vmatrix}. \quad (32)$$

Let us construct transformation changing the matrix on the right to a diagonal form

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g' \\ G' \end{vmatrix} &= \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix}, \\ \begin{vmatrix} g' \\ G' \end{vmatrix} &= S \begin{vmatrix} g \\ G \end{vmatrix}, \quad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}. \end{aligned} \quad (33)$$

The problem to solve is

$$S \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} S^{-1} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix},$$

which results in two linear systems

$$\begin{cases} -\lambda_1 s_{11} - 2iB\gamma s_{12} = 0, \\ 2iB s_{11} - \lambda_1 s_{12} = 0, \end{cases} \quad \begin{cases} -\lambda_2 s_{21} - 2iB\gamma s_{22} = 0, \\ 2iB s_{21} - \lambda_2 s_{22} = 0. \end{cases}$$

The values of λ_1 and λ_2 are given by

$$\lambda_1 = \pm 2B\sqrt{\gamma}, \quad \lambda_2 = \pm 2B\sqrt{\gamma}.$$

The matrix S must be degenerate, so we must use different λ_1, λ_2 :

$$\begin{aligned} \text{Variant (A)} \quad \lambda_1' &= +2B\sqrt{\gamma}, \quad \lambda_2' = -2B\sqrt{\gamma}, \\ i s_{11} - \sqrt{\gamma} s_{12} &= 0, \quad i s_{21} + \sqrt{\gamma} s_{22} = 0; \end{aligned}$$

let it be

$$s_{12} = 1, s_{22} = 1, s_{11} = -i\sqrt{\gamma}, s_{21} = +i\sqrt{\gamma}, \quad S = \begin{vmatrix} -i\sqrt{\gamma} & 1 \\ +i\sqrt{\gamma} & 1 \end{vmatrix}. \quad (34)$$

$$\begin{aligned} \text{Variant (B)} \quad \lambda_1'' &= -2B\sqrt{\gamma} = \lambda_2', \quad \lambda_2'' = +2B\sqrt{\gamma} = \lambda_1', \\ i s_{11} + \sqrt{\gamma} s_{12} &= 0, \quad i s_{21} - \sqrt{\gamma} s_{22} = 0; \end{aligned}$$

let it be

$$s_{12} = 1, s_{22} = 1, s_{11} = +i\sqrt{\gamma}, s_{21} = -i\sqrt{\gamma}, \quad S = \begin{vmatrix} +i\sqrt{\gamma} & 1 \\ -i\sqrt{\gamma} & 1 \end{vmatrix}. \quad (35)$$

In the new (primed) basis, eqs. (32) take the form of two separated differential equations

$$\begin{aligned} (A) \quad & \left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma} \right) g' = 0, \\ & \left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma} \right) G' = 0; \end{aligned} \quad (36)$$

$$\begin{aligned} (B) \quad & \left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma} \right) g'' = 0, \\ & \left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma} \right) G'' = 0. \end{aligned} \quad (37)$$

Recalling the meaning of Δ , let us detail the second order equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0 ,$$

$$\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B \sqrt{\gamma}, \quad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M} . \quad (38)$$

It is convenient to introduce a new variable $x = Br^2$, then eq. (38) reads ¹

$$x \frac{d^2 \varphi}{dx^2} + \frac{d\varphi}{dx} - \left(\frac{m^2}{4x} + \frac{x}{4} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) \varphi = 0 . \quad (39)$$

With the substitution $\varphi(x) = x^A e^{-Cx} f(x)$, for $f(x)$ we get

$$x \frac{d^2 f}{dx^2} + (2A + 1 - 2Cx) \frac{df}{dx} + \left[\frac{A^2 - m^2/4}{x} + (C^2 - \frac{1}{4})x - 2AC - C - \frac{m}{2} + \frac{\lambda^2}{4B} \right] f = 0 .$$

When A, C are taken as $A = + |m|/2$, $C = +1/2$ the previous equation becomes simpler

$$x \frac{d^2 R}{dx^2} + (2A + 1 - x) \frac{dR}{dx} - \left(A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) R = 0 ,$$

which is of (degenerate) hypergeometric type

$$x Y'' + (\gamma - x) Y' - \alpha Y = 0 , \quad \alpha = \frac{|m|}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}, \quad \gamma = |m| + 1 .$$

To obtain polynomials we must impose additional condition $\alpha = -n$; which leads to the following quantization for λ^2

$$\lambda^2 = 4B \left(n + \frac{1}{2} + \frac{|m| + m}{2} \right) . \quad (40)$$

Taking into account (36) – (37), we have relations

$$(A) \left(\Delta + (\epsilon^2 - k^2) - M^2 - 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) g' = 0 , \quad \sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ,$$

$$\left(\Delta + (\epsilon^2 - k^2) - M^2 + 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) G' = 0 ; \quad \sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ;$$

$$(B) \left(\Delta + (\epsilon^2 - k^2) - M^2 + 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) g'' = 0 , \quad \sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ,$$

$$\left(\Delta + (\epsilon^2 - k^2) - M^2 - 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) G'' = 0 ; \quad \sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} .$$

¹For definiteness let us consider B to be positive, which does not affect generality of the analysis. So, to infinite values of r corresponds infinite and positive values of x .

In fact, here there exist only two different possibilities (and correspondingly two formulae for energy spectrum) :

$$\begin{aligned}\sqrt{\epsilon^2 - k^2} &= \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, & q'(r) \neq 0, G' = 0; \\ \sqrt{\epsilon^2 - k^2} &= \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, & q'(r) = 0, G' \neq 0.\end{aligned}\quad (41)$$

In turn, energy spectrum for the case (31) is given by

$$\epsilon^2 = M^2 + k^2 + \lambda^2 \quad (42)$$

Thus, on the base of the use of general covariant formalism in the Petiau – Duffin – Kemmer theory of the vector particle, exact solutions for such a particle are constructed in presence of external homogeneous magnetic field. There are separated three types of linearly independent solutions, and energy spectra are found.

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References

- [1] Rabi I.I. Das freie Electron in Homogenen Magnetfeld nach der Diraschen Theorie. Z. Phys. 1928. Bd. 49. P. 507 – 511.
- [2] Landau L., Diamagnetismus der Metalle, Ztschr. Phys. 1930, Bd. 64, S. 629–637.
- [3] Plesset M.S. Relativistic wave mechanics of the electron deflected by magnetic field. Phys.Rev. 1931. no 12. P. 1728 – 1731.
- [4] I.E. Tamm. Motion of a meson in electromagnetic fields. Collection of papers. Vol. 2. Moskow, Nauka, 1975 95 – 99 (in Russian).
- [5] A.A. Bogush, V.V. Kisel, N.G. Tokarevskaya, V.M. Red'kov. Duffin–Kemmer–Petiau formalism reexamined: non-relativistic approximation for spin 0 and spin 1 particles in a Riemannian space-time. Annales de la Fondation Louis de Broglie. **32**, 355–381 (2007).
- [6] V.M. Red'kov. Fields in Riemannian space and the Lorentz group. Publishing House "Belarusian Science", Minsk (2009).

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The problem of a quantum-mechanical particle in the external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger's) non-relativistic particle with spin 0, and fermions (non-relativistic Pauli's and relativistic Dirac's) with spin 1/2 (the first investigation were [1, 2, 3, 4]). In the present paper, exact solutions for a vector particle with spin 1 will be constructed explicitly. The most popular quantum-mechanical problem for such a particle is that in presence of external Coulomb potential [4].

To treat the problem we take the matrix Petiau – Duffin – Kemmer approach in the theory of the vector particle extended to a general covariant form on the base of tetrad formalism (recent consideration and list of references see in [5, 6]).

The main equation in tetrad form is [6]

$$\left[i \beta^\alpha(x) (\partial_\alpha + B_\alpha - i \frac{e}{\hbar} A_\alpha) - \frac{Mc}{\hbar} \right] \Psi(x) = 0 ,$$

$$\beta^\alpha(x) = \beta^a e_{(a)}^\alpha(x), \quad B_\alpha(x) = \frac{1}{2} J^{ab} e_{(a)}^\beta \nabla_\alpha e_{(b)\beta} ; \quad (1)$$

$e_{(a)}^\alpha(x)$ is a tetrad, J^{ab} stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note Mc/\hbar as M). To the homogeneous magnetic field $\mathbf{B} = (0, 0, B)$ corresponds 4-potential

$$A^a = (0, \vec{A}) = (0, \frac{1}{2} \vec{B} \times \vec{r}) = \frac{B}{2} (0, -x^2, +x^1, 0) ;$$

in the cylindric coordinates it is given by a simple expression

$$(ct, r, \phi, z), \quad dS^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 ,$$

$$A_0 = 0, \quad A_r = 0, \quad A_\phi = -\frac{Br^2}{2}, \quad A_z = 0 . \quad (2)$$

Choosing a diagonal cylindric tetrad

$$e_{(0)}^\alpha = (1, 0, 0, 0), \quad e_{(1)}^\alpha = (0, 1, 0, 0), \quad e_{(2)}^\alpha = (0, 0, \frac{1}{r}, 0), \quad e_{(3)}^\alpha = (0, 0, 0, 1) . \quad (3)$$

after simple calculation, the main equation (1) is reduced to the form

$$\left[i\beta^0 \partial_0 + i\beta^1 \partial_r + i\frac{\beta^2}{r} (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + i\beta^3 \partial_z - M \right] \Psi(t, r, \phi, z) = 0. \quad (4)$$

For brevity we will note $(eB/2\hbar)$ as B . It is best to chose the matrices β^a in the so-called cyclic form, where the generator J^{12} has a diagonal structure. In block-form $(1 - 3 - 3 - 3)$ these matrices are

$$\beta^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \beta^i = \begin{vmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{vmatrix},$$

where e_i, τ_i denote

$$e_1 = \frac{1}{\sqrt{2}}(-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad e_3 = (0, i, 0),$$

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{vmatrix}, \quad \tau_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = s_3.$$

Entering eq. (4) generator J^{12} is given by

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 = -i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{vmatrix} = -i S_3.$$

2 Separation of the variables

With the use of special substitution (it corresponds to diagonalization of the third projections of momentum P_3 and angular momentum J_3 for a particle with spin 1)

$$\Psi = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix}, \quad \left[\epsilon\beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M \right] \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix} = 0. \quad (5)$$

after calculations we arrive at the radial system of ten equations

$$\begin{aligned} -b_{m-1} E_1 - a_{m+1} E_3 - ik E_2 &= M\Phi_0, \\ -ib_{m-1} H_1 + ia_{m+1} H_3 + i\epsilon E_2 &= M\Phi_2, \\ ia_m H_2 + i\epsilon E_1 - k H_1 &= M\Phi_1, \\ -ib_m H_2 + i\epsilon E_3 + k H_3 &= M\Phi_3, \end{aligned} \quad (6)$$

$$\begin{aligned} a_m \Phi_0 - i\epsilon \Phi_1 &= ME_1, & -ia_m \Phi_2 + k \Phi_1 &= MH_1, \\ b_m \Phi_0 - i\epsilon \Phi_3 &= ME_3, & ib_m \Phi_2 - k \Phi_3 &= MH_3, \\ -i\epsilon \Phi_2 - ik \Phi_0 &= ME_2, & ib_{m-1} \Phi_1 - ia_{m+1} \Phi_3 &= MH_2, \end{aligned} \quad (7)$$

where special abbreviations are used for first order differential operators

$$\frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m + Br^2}{r}\right) = a_m, \quad \frac{1}{\sqrt{2}}\left(-\frac{d}{dr} + \frac{m + Br^2}{r}\right) = b_m.$$

From (6) – (7) it follow 4 equations for the components Φ_a

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 + i\epsilon (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) &= 0, \\ (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 - ik (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) &= 0, \\ (-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i\epsilon a_m \Phi_0 + ik a_m \Phi_2 &= 0, \\ (-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i\epsilon b_m \Phi_0 + ik b_m \Phi_2 &= 0; \end{aligned} \quad (8)$$

3 Special simple class of solutions

There exists a simple linear condition on 4-vector Φ_a , leading to a second order differential equation. Let it be $\Phi_1 = 0$, $\Phi_3 = 0$, the system (8) gives

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 &= 0, \\ (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 &= 0, \\ i a_m (\epsilon \Phi_0 + ik \Phi_2) = 0, \quad i b_m (\epsilon \Phi_0 + ik \Phi_2) &= 0. \end{aligned} \quad (9)$$

From two last equations in (9) we conclude that

$$\epsilon \Phi_0 + k \Phi_2 = 0 \quad (10)$$

correspondingly, the first two in (9) take the form

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2) \Phi_0 &= 0, \\ (-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2) \Phi_2 &= 0. \end{aligned} \quad (11)$$

Because, we can readily get

$$-b_{m-1} a_m - a_{m+1} b_m = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} = \Delta,$$

eqs. (11) are differential equations of one the same type that is operative in the theory of a scalar particle in magnetic field

$$(\Delta + \epsilon^2 - k^2 - M^2) \Phi_0 = 0, \quad (\Delta + \epsilon^2 - k^2 - M^2) \Phi_2 = 0. \quad (12)$$

All the remaining component of the 10-dimensional function can be found straightforwardly as in accordance with the relations

$$\begin{aligned} \Phi_1 = 0, \quad \Phi_3 = 0, \quad \epsilon \Phi_0 + k \Phi_2 &= 0, \\ a_m \Phi_0 = ME_1, \quad a_m \Phi_2 = iMH_1, \quad b_m \Phi_0 = ME_3, \\ b_m \Phi_2 = -iMH_3, \quad (\epsilon \Phi_2 + k \Phi_0) = iME_2, \quad 0 = H_2. \end{aligned} \quad (13)$$

In general, there must exist three types of solutions for the particle with spin 1, we have found only one that.

4 General analysis of the radial equations

Eqs. (8) can be transformed to the form

$$\begin{aligned} & [-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2] (k \Phi_0 + \epsilon \Phi_2) = 0 , \\ & [-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2] (\epsilon \Phi_0 + k \Phi_2) = \\ & = (\epsilon^2 - k^2) [(\epsilon \Phi_0 + k \Phi_2) - (i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3)] ; \end{aligned} \quad (14)$$

$$\begin{aligned} & (-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i \epsilon a_m \Phi_0 + i k a_m \Phi_2 = 0 , \\ & (-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i \epsilon b_m \Phi_0 + i k b_m \Phi_2 = 0 . \end{aligned} \quad (15)$$

Let us introduce new variables

$$F(r) = k \Phi_0(r) + \epsilon \Phi_2(r) , \quad G(r) = \epsilon \Phi_0(r) + k \Phi_2(r) , \quad (16)$$

then eqs. (14) – (15) read

$$\begin{aligned} & [-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2] F = 0 , \\ & [-b_{m-1} a_m - a_{m+1} b_m - M^2] G = -(\epsilon^2 - k^2) (i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3) , \end{aligned} \quad (17)$$

$$\begin{aligned} & (-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i a_m G = 0 , \\ & (-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i b_m G = 0 . \end{aligned} \quad (18)$$

For two equations in (18), let us multiply the first (from the left) by b_{m-1} and the second by the a_{m+1} , which result in

$$\begin{aligned} & -b_{m-1} a_m (b_{m-1} \Phi_1) + (\epsilon^2 - k^2 - M^2) (b_{m-1} \Phi_1) + b_{m-1} a_m (a_{m+1} \Phi_3) + i b_{m-1} a_m G = 0 , \\ & -a_{m+1} b_m (a_{m+1} \Phi_3) + (\epsilon^2 - M^2 - k^2) (a_{m+1} \Phi_3) + a_{m+1} b_m (b_{m-1} \Phi_1) + i a_{m+1} b_m G = 0 . \end{aligned} \quad (19)$$

Again, let us introduce two new variables

$$b_{m-1} \Phi_1 = Z_1 , \quad a_{m+1} \Phi_3 = Z_3 ; \quad (20)$$

eqs. (19) read as follows

$$\begin{aligned} & -b_{m-1} a_m Z_1 + (\epsilon^2 - k^2 - M^2) Z_1 + b_{m-1} a_m Z_3 + i b_{m-1} a_m G = 0 , \\ & -a_{m+1} b_m Z_3 + (\epsilon^2 - M^2 - k^2) Z_3 + a_{m+1} b_m Z_1 + i a_{m+1} b_m G = 0 . \end{aligned} \quad (21)$$

With the aid of new functions $f(r), g(r)$

$$Z_1 = \frac{f+g}{2} , \quad Z_3 = \frac{f-g}{2} , \quad Z_1 + Z_3 = f , \quad Z_1 - Z_3 = g ; \quad (22)$$

the system (21) is transformed to the following ones

$$\begin{aligned} & -b_{m-1} a_m g + (\epsilon^2 - k^2 - M^2) \frac{f+g}{2} + i b_{m-1} a_m G = 0 , \\ & a_{m+1} b_m g + (\epsilon^2 - M^2 - k^2) \frac{f-g}{2} + i a_{m+1} b_m G = 0 . \end{aligned} \quad (23)$$

Combining these equations we get

$$\begin{aligned} [-b_{m-1}a_m - a_{m+1}b_m + \epsilon^2 - k^2 - M^2] g + i(b_{m-1}a_m - a_{m+1}b_m) G &= 0, \\ (-b_{m-1}a_m + a_{m+1}b_m) g + (\epsilon^2 - k^2 - M^2)f + i(b_{m-1}a_m + a_{m+1}b_m)G &= 0. \end{aligned} \quad (24)$$

In these variables, eqs. (17) can be written as

$$\begin{aligned} (-b_{m-1} a_m - a_{m+1}b_m + \epsilon^2 - M^2 - k^2) F &= 0, \\ (-b_{m-1}a_m - a_{m+1}b_m - M^2) G &= -i(\epsilon^2 - k^2) f. \end{aligned} \quad (25)$$

Further, with the use of identities

$$-b_{m-1} a_m - a_{m+1}b_m = \Delta, \quad -b_{m-1} a_m + a_{m+1}b_m = 2B. \quad (26)$$

eqs. (25) and (24) can be written down as follows

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) F &= 0, \\ \Delta G &= M^2 G - i(\epsilon^2 - k^2) f, \\ (\Delta + \epsilon^2 - k^2 - M^2) g &= 2iB G, \\ (\epsilon^2 - k^2 - M^2) f - i\Delta G + 2B g &= 0. \end{aligned} \quad (27)$$

With the help of the second equation, from the forth one it follows the linear relationship

$$f = -i G + \frac{2B}{M^2} g. \quad (28)$$

Now, excluding the function f in the second one in (27)

$$(\Delta + \epsilon^2 - k^2 - M^2) G = -i(\epsilon^2 - k^2) \frac{2B}{M^2} g. \quad (29)$$

Thus, the general problem is reduced to the system of four equations

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) F &= 0, \\ f &= -i G + \frac{2B}{M^2} g, \\ (\Delta + \epsilon^2 - k^2 - M^2) g &= 2iB G, \\ (\Delta + \epsilon^2 - k^2 - M^2) G &= -2iB \frac{\epsilon^2 - k^2}{M^2} g, \end{aligned} \quad (30)$$

The structure of this system allows to separate an evident linearly independent solution as follows

$$\begin{aligned} f(r) &= 0, & g(r) &= 0, & H(r) &= 9, \\ F(r) &\neq 0, & (\Delta - k^2 - M^2 + \epsilon^2) F &= 0. \end{aligned} \quad (31)$$

corresponding functions and energy spectrum are known (also see below). We are to solve the system of two last equations in (30), in matrix form it reads (let $\gamma = (\epsilon^2 - k^2)/M^2$)

$$(\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g(r) \\ G(r) \end{vmatrix} = \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} \begin{vmatrix} g(r) \\ G(r) \end{vmatrix}. \quad (32)$$

Let us construct transformation changing the matrix on the right to a diagonal form

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g' \\ G' \end{vmatrix} &= \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix}, \\ \begin{vmatrix} g' \\ G' \end{vmatrix} &= S \begin{vmatrix} g \\ G \end{vmatrix}, \quad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}. \end{aligned} \quad (33)$$

The problem to solve is

$$S \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} S^{-1} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix},$$

which results in two linear systems

$$\begin{cases} -\lambda_1 s_{11} - 2iB\gamma s_{12} = 0, \\ 2iB s_{11} - \lambda_1 s_{12} = 0, \end{cases} \quad \begin{cases} -\lambda_2 s_{21} - 2iB\gamma s_{22} = 0, \\ 2iB s_{21} - \lambda_2 s_{22} = 0. \end{cases}$$

The values of λ_1 and λ_2 are given by

$$\lambda_1 = \pm 2B\sqrt{\gamma}, \quad \lambda_2 = \pm 2B\sqrt{\gamma}.$$

The matrix S must be degenerate, so we must use different λ_1, λ_2 :

$$\begin{aligned} \text{Variant (A)} \quad \lambda_1' &= +2B\sqrt{\gamma}, \quad \lambda_2' = -2B\sqrt{\gamma}, \\ i s_{11} - \sqrt{\gamma} s_{12} &= 0, \quad i s_{21} + \sqrt{\gamma} s_{22} = 0; \end{aligned}$$

let it be

$$s_{12} = 1, s_{22} = 1, s_{11} = -i\sqrt{\gamma}, s_{21} = +i\sqrt{\gamma}, \quad S = \begin{vmatrix} -i\sqrt{\gamma} & 1 \\ +i\sqrt{\gamma} & 1 \end{vmatrix}. \quad (34)$$

$$\begin{aligned} \text{Variant (B)} \quad \lambda_1'' &= -2B\sqrt{\gamma} = \lambda_2', \quad \lambda_2'' = +2B\sqrt{\gamma} = \lambda_1', \\ i s_{11} + \sqrt{\gamma} s_{12} &= 0, \quad i s_{21} - \sqrt{\gamma} s_{22} = 0; \end{aligned}$$

let it be

$$s_{12} = 1, s_{22} = 1, s_{11} = +i\sqrt{\gamma}, s_{21} = -i\sqrt{\gamma}, \quad S = \begin{vmatrix} +i\sqrt{\gamma} & 1 \\ -i\sqrt{\gamma} & 1 \end{vmatrix}. \quad (35)$$

In the new (primed) basis, eqs. (32) take the form of two separated differential equations

$$\begin{aligned} (A) \quad & \left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma} \right) g' = 0, \\ & \left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma} \right) G' = 0; \end{aligned} \quad (36)$$

$$\begin{aligned} (B) \quad & \left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma} \right) g'' = 0, \\ & \left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma} \right) G'' = 0. \end{aligned} \quad (37)$$

Recalling the meaning of Δ , let us detail the second order equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0 ,$$

$$\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B \sqrt{\gamma}, \quad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M} . \quad (38)$$

It is convenient to introduce a new variable $x = Br^2$, then eq. (38) reads ¹

$$x \frac{d^2 \varphi}{dx^2} + \frac{d\varphi}{dx} - \left(\frac{m^2}{4x} + \frac{x}{4} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) \varphi = 0 . \quad (39)$$

With the substitution $\varphi(x) = x^A e^{-Cx} f(x)$, for $f(x)$ we get

$$x \frac{d^2 f}{dx^2} + (2A + 1 - 2Cx) \frac{df}{dx} + \left[\frac{A^2 - m^2/4}{x} + (C^2 - \frac{1}{4})x - 2AC - C - \frac{m}{2} + \frac{\lambda^2}{4B} \right] f = 0 .$$

When A, C are taken as $A = + |m|/2$, $C = +1/2$ the previous equation becomes simpler

$$x \frac{d^2 R}{dx^2} + (2A + 1 - x) \frac{dR}{dx} - \left(A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) R = 0 ,$$

which is of (degenerate) hypergeometric type

$$x Y'' + (\gamma - x) Y' - \alpha Y = 0 , \quad \alpha = \frac{|m|}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}, \quad \gamma = |m| + 1 .$$

To obtain polynomials we must impose additional condition $\alpha = -n$; which leads to the following quantization for λ^2

$$\lambda^2 = 4B \left(n + \frac{1}{2} + \frac{|m| + m}{2} \right) . \quad (40)$$

Taking into account (36) – (37), we have relations

$$(A) \left(\Delta + (\epsilon^2 - k^2) - M^2 - 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) g' = 0 , \quad \sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ,$$

$$\left(\Delta + (\epsilon^2 - k^2) - M^2 + 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) G' = 0 ; \quad \sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ;$$

$$(B) \left(\Delta + (\epsilon^2 - k^2) - M^2 + 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) g'' = 0 , \quad \sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ,$$

$$\left(\Delta + (\epsilon^2 - k^2) - M^2 - 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) G'' = 0 ; \quad \sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} .$$

¹For definiteness let us consider B to be positive, which does not affect generality of the analysis. So, to infinite values of r corresponds infinite and positive values of x .

In fact, here there exist only two different possibilities (and correspondingly two formulae for energy spectrum) :

$$\begin{aligned}\sqrt{\epsilon^2 - k^2} &= \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, & q'(r) \neq 0, G' = 0 ; \\ \sqrt{\epsilon^2 - k^2} &= \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, & q'(r) = 0, G' \neq 0 .\end{aligned}\quad (41)$$

In turn, energy spectrum for the case (31) is given by

$$\epsilon^2 = M^2 + k^2 + \lambda^2 \quad (42)$$

Thus, on the base of the use of general covariant formalism in the Petiau – Duffin – Kemmer theory of the vector particle, exact solutions for such a particle are constructed in presence of external homogeneous magnetic field. There are separated three types of linearly independent solutions, and energy spectra are found.

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References

- [1] Rabi I.I. Das freie Electron in Homogenen Magnetfeld nach der Diraschen Theorie. Z. Phys. 1928. Bd. 49. P. 507 – 511.
- [2] Landau L., Diamagnetismus der Metalle, Ztschr. Phys. 1930, Bd. 64, S. 629–637.
- [3] Plesset M.S. Relativistic wave mechanics of the electron deflected by magnetic field. Phys.Rev. 1931. no 12. P. 1728 – 1731.
- [4] I.E. Tamm. Motion of a meson in electromagnetic fields. Collection of papers. Vol. 2. Moskow, Nauka, 1975 95 – 99 (in Russian).
- [5] A.A. Bogush, V.V. Kisel, N.G. Tokarevskaya, V.M. Red'kov. Duffin–Kemmer–Petiau formalism reexamined: non-relativistic approximation for spin 0 and spin 1 particles in a Riemannian space-time. Annales de la Fondation Louis de Broglie. **32**, 355–381 (2007).
- [6] V.M. Red'kov. Fields in Riemannian space and the Lorentz group. Publishing House "Belarusian Science", Minsk (2009).